1 Week 1

1.1 The symmetric group - August 1, 2025

A permutation of n, or a permutation of $\{1, \ldots, n\}$ is a $w \in \mathcal{M}_n(\mathbb{C})$ such that

1. There is exactly one non-zero entry in each row and column
2. The non-zero entries are 1.

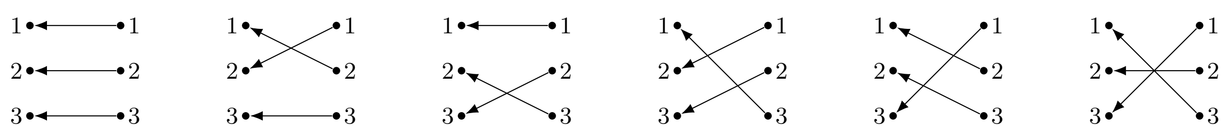
Alternatively interchange 1 with the roots of unity $z^n = 1$ to give other groups similar to the symmetric group.

$$S_n = \{w \in \mathcal{M}_n(\mathbb{C}) \mid w \text{ is a permutation of } \{1, \ldots, n\}\}.$$

S_3 is therefore the matrices:

$$S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & \end{pmatrix}$$

The matrices of S_3 can be expressed as graphs:



Identify a permutation for some $w \in S_n$ with the bijection

$$w : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$$

given by $w(i) = j$ if $w_{ji} = 1$, where $w_{i,j}$ is the (i, j) entry in an $n \times n$ matrix.

Think of matrices as functions. A $n \times n$ matrix A with entries in $\mathbb{C}$ is a function,

$$\begin{aligned} A \colon \{1, \ldots, n\} \times \{1, \ldots, n\} &\to \mathbb{C} \\ (i, j) &\mapsto A_{ij} \end{aligned}$$

Multplication of graphs is as follows: